

On dp-minimal fields

Will Johnson

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Abstract

We classify dp-minimal pure fields up to elementary equivalence. Most are equivalent to Hahn series fields $K((t^\Gamma))$ where Γ satisfies some divisibility conditions and K is \mathbb{F}_p^{alg} or a local field of characteristic zero. We show that dp-small fields (including VC-minimal fields) are algebraically closed or real closed.

1 Introduction

A theory is said to be *VC-minimal* if there is a family \mathcal{F} of subsets of the home sort such that

- The family \mathcal{F} is ind-definable without parameters.
- Every definable subset of the home sort is a boolean combination of sets in \mathcal{F} .
- If $X, Y \in \mathcal{F}$, then either $X \subseteq Y$, $Y \subseteq X$, or $X \cap Y = \emptyset$.

This definition is due to Adler [1]. Strongly minimal, C-minimal, o-minimal, and weakly o-minimal theories are all VC-minimal.

A theory is *not dp-minimal* if there is a model M and formulas $\phi(x; y)$, $\psi(x; z)$ with $|x| = 1$, and elements a_{ij}, b_i, c_j such that for all i, j, i', j' ,

$$\begin{aligned} i = i' &\iff M \models \phi(a_{ij}, b_{i'}) \\ j = j' &\iff M \models \psi(a_{ij}, c_{j'}) \end{aligned}$$

Otherwise, it is said to be dp-minimal. Dp-minimality first appeared in Shelah [13] and was isolated as an interesting concept by Onshuus and Usvyatsov [11]. It is known that VC-minimal theories and p -minimal theories are dp-minimal—see [2]. Dp-minimality is equivalent to having dp-rank 1; we will discuss dp-rank in §2.1 below.

VC-minimality is not preserved under reducts, but dp-minimality is.

Here are our main results, which essentially classify dp-minimal fields up to elementary equivalence as pure fields.

Theorem 1.1. *Let (K, v) be a henselian defectless valued field, with residue field k and value group Γ (possibly trivial). Suppose*

- $k \models ACF_p$ or k is elementarily equivalent to a local field of characteristic 0.

- For every n , $|\Gamma/n\Gamma|$ is finite
- If k has characteristic p , and $\gamma \in [-v(p), v(p)]$ then $\gamma \in p \cdot \Gamma$. Here $[-v(p), v(p)]$ denotes $\Gamma = (-\infty, \infty)$ if K has characteristic p .

Then (K, v) is dp-minimal as a valued field, and the theory of (K, v) is completely determined by the theories of k and Γ (or k and $(\Gamma, v(p))$ in mixed characteristic).

The surprising result is that all pure dp-minimal fields arise this way.

Theorem 1.2. *Let K be a sufficiently saturated dp-minimal field. Then there is some valuation on K satisfying the conditions of Theorem 1.1.*

This *almost* says that all dp-minimal fields are elementarily equivalent to ones of the form $K((t^\Gamma))$ where K is \mathbb{F}_p^{alg} or a characteristic zero local field, and Γ satisfies some divisibility conditions. The one exceptional case is the mixed characteristic case, which includes fields such as the spherical completion of $\mathbb{Z}_p^{un}(p^{1/p^\infty})$.

To obtain a very precise classification of pure dp-minimal fields, we would need to determine which (K, v) as in Theorem 1.1 are elementarily equivalent as pure fields. (For example, $\mathbb{C}((t^\mathbb{Z}))$ is elementarily equivalent as a pure field to $\mathbb{C}((t^{\mathbb{Z} \times \mathbb{Q}}))$, even though \mathbb{Z} and $\mathbb{Z} \times \mathbb{Q}$ are not elementarily equivalent as ordered groups.) We will not do this here—though it seems highly likely that extraneous factors of \mathbb{Q} are the only thing that can go wrong.

From the above results, we will obtain the following corollary in §7.

Theorem 1.3. *Let K be a VC-minimal field. Then K is algebraically closed or real closed.*

As we will see, this holds if “VC-minimal” is replaced with Guingona’s notion of “dp-small” [4].

1.1 Previous work on dp-minimal fields

Dolich, Goodrick, and Lippel showed that \mathbb{Q}_p is dp-minimal [2]. John Goodrick [3] and Pierre Simon [14] proved some results concerning divisible ordered dp-minimal groups: Goodrick proved an analogue of the monotonicity theorem for o-minimal structures, and Simon proved that infinite sets have non-empty interior. Building off their work, as well as [10], Vince Guingona proved that VC-minimal ordered fields are real closed [4].

Very recently, Walsberg, Jahnke, and Simon have classified dp-minimal *ordered* fields [7], among other things. In fact, they have independently obtained many of the results described below—they essentially proved our main result Theorem 1.2 *modulo* an assumption, which is essentially Theorem 4.16 below (see Propositions 7.4 and 8.1 in [7]).

1.2 Outline

We will focus on Theorem 1.2, the truly interesting result. Theorem 1.1 is an exercise in quantifier elimination, though we will include a proof sketch in §8 below.

In §3 through §5, we will focus on dp-minimal fields which are not strongly minimal. The upshot of this will be the fact that they admit t-henselian V-topologies (essentially Theorem 5.13). In §6.2 we will apply results of Jahnke and Koenigsmann [6] to pick out the desired valuation. Finally, in §6.3 we will obtain the divisibility and defectlessness conditions using results of Kaplan-Scanlon-Wagner [9].

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2 Background material

We review some necessary background material on dp-rank §2.1 and field topologies §2.2.

2.1 Dp-rank

If X is a type-definable set and κ is a cardinal, a *randomness pattern of depth κ in X* is a collection of formulas $\{\phi_\alpha(x; y_\alpha) : \alpha < \kappa\}$ and elements $\{b_{i,j} : i < \kappa, j < \omega\}$ such

that for every function $\eta : \kappa \rightarrow \omega$ there is some element a_η in X such that for all i, j

$$j = \eta(i) \iff \phi_i(a_\eta, b_{ij})$$

The dp-rank of X is defined to be the supremum of the cardinals κ such that there is a randomness pattern of depth κ in X . This definition first appears in [16].

The following fundamental facts about dp-rank are either easy, or proven in [8].

1. The formula $x = x$ has dp-rank less than ∞ if and only if the theory is NIP.
2. The formula $x = x$ has dp-rank 1 if and only if the theory is dp-minimal.
3. If X is type-definable over A , then $\text{dp-rk}(X)$ is the supremum of $\text{dp-rk}(x/A)$ for $x \in X$.
4. $\text{dp-rk}(X) > 0$ if and only if X is infinite.
5. For $n < \omega$, $\text{dp-rk}(a/A) \geq n$ if and only if there are sequences I_1, \dots, I_n , which are mutually indiscernible over A , such that each sequence is not individually Aa -indiscernible.
6. Dp-rank is subadditive: $\text{dp-rk}(ab/A) \leq \text{dp-rk}(a/bA) + \text{dp-rk}(b/A)$.
7. If X and Y are non-empty type-definable sets, then $\text{dp-rk}(X \times Y) = \text{dp-rk}(X) + \text{dp-rk}(Y)$.
8. If $\text{dp-rk}(a/A) = n$ and X is an A -definable set of dp-rank 1, then there is $b \in X$ such that $\text{dp-rk}(ab/A) = n + 1$.
9. If $X \twoheadrightarrow Y$ is a definable surjection, then $\text{dp-rk}(Y) \leq \text{dp-rk}(X)$.

Here are some basic uses of dp-rank:

Observation 2.1. *Let K be a field of finite dp-rank. Then K is perfect.*

Proof. The field K^p of p th powers is in definable bijection with K , so it has the same rank as K . If K is imperfect, then K is a definable K^p vector space of dimension greater than 1. It contains a two-dimensional subspace, so $K^p \times K^p$ injects definably into K . This shows

$$\text{dp-rk}(K) \geq 2 \cdot \text{dp-rk}(K^p) = 2 \cdot \text{dp-rk}(K)$$

So $\text{dp-rk}(K) = 0$, and K is finite. Finite fields are perfect. \square

Observation 2.2. *Let K be dp-minimal field. Then K eliminates \exists^∞ (in powers of the home sort).*

Proof. It suffices to show that a definable set $X \subset K$ is finite if and only if there is some $a \in K$ such that the map $(x, y) \mapsto x + a \cdot y$ is injective on $X \times X$. If X is finite, any a outside the finite set

$$\left\{ \frac{x_1 - x_2}{x_3 - x_4} : \vec{x} \in X^4 \right\}$$

will work. If X is infinite, then $\text{dp-rk}(X) \geq 1$, so $\text{dp-rk}(X \times X) = 2$ and $X \times X$ cannot definably inject into K . \square

This has the following useful corollary:

Corollary 2.3. *Suppose K is dp-minimal. Then any infinite externally definable subset of K contains an infinite internally definable set.*

Proof. Suppose $S \subset K$ is externally definable. By honest definitions ([15] Remark 3.14), there is some formula $\phi(x; y)$ such that for every finite $S_0 \subset S$, there is $b \in K$ such that $S_0 \subset \phi(K; b) \subset S$. By elimination of \exists^∞ , there is some number n such that $\phi(K; b)$ is infinite or has size less than n . If we choose S_0 to have size greater than n , then $\phi(K; b)$ will be our desired infinite internally-definable set. \square

2.2 Filters and topologies

Let K be a field, and τ be a family of subsets of K that is *filtered*, in the sense that $\forall U, V \in \tau \exists W \in \tau : W \subset U \cap V$. Consider the following conditions on τ , lifted straight out of [12]. Here, set-quantifiers range over τ and element-quantifiers range over K :

1. $\forall U : \{0\} \subsetneq U$
2. $\forall x \neq 0 \exists U : x \notin U$
3. $\forall U \exists V : V - V \subset U$
4. $\forall U, x \exists V : x \cdot V \subseteq U$
5. $\forall U \exists V : V \cdot V \subseteq U$
6. $\forall U \exists V : (1 + V)^{-1} \subseteq (1 + U)$
7. $\forall U \exists V \forall x, y : (x \cdot y \in V \implies (x \in U \vee y \in U))$

Then τ is a neighborhood basis for a Hausdorff non-discrete group topology on $(K, +)$ if and only if conditions 1-3 hold, and the topology is uniquely determined in this case. The topology is a ring topology if and only if conditions 1-5 hold, a field topology if and only if 1-6 hold, and a V-topology if and only if 1-7 hold.

If τ and τ' are two different filtered families on K satisfying 1-3, then τ and τ' will define the same topology if and only if the following two conditions hold:

- For all $U \in \tau$ there is $V \in \tau'$ such that $V \subseteq U$.
- For all $U \in \tau'$ there is $V \in \tau$ such that $V \subseteq U$.

Now suppose that K is a field with some additional structure, and the sets in τ are all definable. In a saturated elementary extension $\mathbb{C} \succeq K$, the intersection of the sets in τ is a set $I_\tau \subset \mathbb{C}$ that is type-definable over K . The conditions above all translate into conditions on I_τ :

1. $I_\tau \supsetneq \{0\}$
2. I_τ does not intersect K^\times .
3. I_τ is a subgroup of $(\mathbb{C}, +)$
4. I_τ is closed under multiplication by elements of K
5. I_τ is closed under multiplication

6. $(1 + I_\tau)^{-1} = (1 + I_\tau)$
7. $\mathbb{C} \setminus I_\tau$ is closed under multiplication

respectively. Moreover, τ and τ' induce the same topology if and only if $I_\tau = I_{\tau'}$. So we have an injective map from type-definable sets over K satisfying conditions 1-3, to topologies on K .

We will use the following two observations later.

Observation 2.4. *Conditions 1-5 and 7 together imply condition 6, and that I_τ is the maximal ideal of a valuation ring \mathcal{O} containing K .*

Proof. Suppose conditions 1-5 and 7 hold. Let \mathcal{O} be the set of $x \in \mathbb{C}$ such that $x \cdot I_\tau \subseteq I_\tau$. This is obviously closed under multiplication, and is closed under addition and subtraction by condition 3. So it is a subring of \mathbb{C} . It contains I_τ by condition 5, and so I_τ is an ideal in \mathcal{O} . It is a proper ideal by condition 2. Also, \mathcal{O} contains K by condition 4.

If $x \in \mathbb{C} \setminus I_\tau$, then multiplication by x preserves the complement of I_τ , so division by x preserves I_τ . Thus

$$x \notin I_\tau \implies 1/x \in \mathcal{O}$$

Equivalently, $1/x \notin \mathcal{O} \implies x \in I_\tau$. This ensures that \mathcal{O} is a valuation ring with maximal ideal I_τ .

Now it is a general fact that if \mathfrak{m} is the maximal ideal of a valuation ring, then $(1 + \mathfrak{m})^{-1} = 1 + \mathfrak{m}$, so condition 6 holds. \square

Observation 2.5. *Suppose τ and τ' satisfy conditions 1-4 and that $a \cdot I_\tau \subseteq I_{\tau'}$ for some $a \in \mathbb{C}^\times$. Then $I_\tau \subseteq I_{\tau'}$.*

Proof. For $U \in \tau'$, we have $a \cdot I_\tau \subseteq I_{\tau'} \subseteq U$. By compactness, there is some $V \in \tau$ such that $a \cdot V \subseteq U$. As $K \preceq \mathbb{C}$, there is some $a' \in K$ such that $a' \cdot V \subseteq U$. By condition 4 on τ ,

$$I_\tau = a' \cdot I_\tau \subseteq a' \cdot V \subseteq U$$

As U was arbitrary, $I_\tau \subseteq I_{\tau'}$. \square

3 Infinitesimals

Until §6, let \mathbb{C} be a fairly saturated dp-minimal field **that is not strongly minimal**.

If $X, Y \subset \mathbb{C}$, let $X -_\infty Y$ denote

$$\{c \in \mathbb{C} : \exists^\infty y \in Y : c + y \in X\}$$

This is a subset of $X - Y$. It is definable if X and Y are, by Observation 2.2.

Lemma 3.1. *If X and Y are infinite, so is $X -_\infty Y$.*

Proof. Suppose X and Y are A -definable. Take $(x, y) \in X \times Y$ of dp-rank 2 over A , and let $c = x - y$. By subadditivity of dp-rank, and dp-minimality,

$$2 = \text{dp-rk}(x, y/A) = \text{dp-rk}(y, c/A) \leq \text{dp-rk}(y/c, A) + \text{dp-rk}(c/A) \leq 1 + 1$$

Equality must hold, so $y \notin \text{acl}(Ac)$ and $c \notin \text{acl}(A)$. As $y \in Y \cap (X - c)$, the Ac -definable set $Y \cap (X - c)$ is infinite. Then $c \in X -_\infty Y$, so the A -definable set $X -_\infty Y$ is infinite. \square

Proposition 3.2. *Let $K \preceq \mathbb{C}$ be a small model. Let*

$$\tau = \{X -_\infty X : X \subset K \text{ is infinite and } K\text{-definable}\}$$

Then τ is a filtered family on K and it satisfies conditions 1, 2, 4 of §2.2:

1. $\forall U \in \tau : \{0\} \subsetneq U$
2. $\forall x \neq 0, \exists U \in \tau : x \notin U$
4. $\forall U \in \tau, \forall x, \exists V \in \tau : x \cdot V \subseteq U$

Proof. To see τ is filtered, suppose X and Y are infinite sets. As $X -_\infty Y$ is non-empty, there is some translate Y' of Y such that $X \cap Y'$ is infinite. Then $Y' -_\infty Y' = Y -_\infty Y$ and

$$(X \cap Y') -_\infty (X \cap Y') \subseteq (X -_\infty X) \cap (Y' -_\infty Y') = (X -_\infty X) \cap (Y -_\infty Y)$$

so τ is filtered.

Condition 1 follows because $0 \in X -_\infty X$ for any infinite X , and $X -_\infty X$ is infinite by the lemma.

Something slightly stronger than condition 4 is true: if $U \in \tau$, and $a \in K^\times$, then $a \cdot U \in \tau$. This follows from the identity:

$$(a \cdot X) -_\infty (a \cdot X) = a \cdot (X -_\infty X)$$

In light of this, condition 2 reduces to showing that $X -_\infty X \neq K$ for some infinite X . By failure of strong minimality and Observation 2.2, there is a K -definable set D which is infinite and co-infinite. Let D' be the complement of D . By the Lemma, $D -_\infty D'$ is non-empty, so there is some c such that $X := D \cap (D' + c)$ is infinite. Then $X - c \subseteq D'$ so $(X - c) \cap X = \emptyset$, and $c \notin X -_\infty X$. \square

We'll denote the corresponding type-definable set by I_K , and refer to elements as K -infinitesimals. So $\epsilon \in \mathbb{C}$ is K -infinitesimal if and only if $\epsilon \in X -_\infty X$ for every infinite K -definable set $X \subset \mathbb{C}$. Equivalently, $X \cap (X - \epsilon)$ is infinite for every infinite K -definable set X .

Conditions 1, 2, and 4 of Proposition 3.2 translate into the following facts: $0 \subsetneq I_K$, $I_K \cap K = \{0\}$, and I_K is closed under multiplication by K .

3.1 Sums of infinitesimals

In this section we show that I_K is closed under addition and subtraction, (condition 3 of §2.2), which ensures that τ is a neighborhood basis of a group topology on the additive group.

Definition 3.3. *Let K be a small model. A \mathbb{C} -definable bijection $f : \mathbb{C} \rightarrow \mathbb{C}$ is K -slight if $X \cap f^{-1}(X)$ is infinite for every K -definable infinite set X .*

For example, the translation map $x \mapsto x + \epsilon$ is K -slight if and only if ϵ is a K -infinitesimal.

The main goal here is to show that K -slight maps form a group under composition.

Definition 3.4. *Let K be a small model, and $X \subset \mathbb{C}$ be K -definable. Say that a \mathbb{C} -definable bijection f “ K -displaces X ” if $X(K) \cap f^{-1}(X)$ is empty.*

Lemma 3.5. *Suppose $K' \succeq K$ and f' and f are \mathbb{C} -definable bijections such that $\text{tp}(f'/K')$ is an heir of $\text{tp}(f/K)$. (Here, we are identifying a bijection with its code.)*

- *If f is K -slight, then f' is K -slight.*
- *If X is K -displaced by f , then X is K' -displaced by f' .*

Proof. First suppose f is K -slight. As $f' \equiv_K f$, the map f' is also K -slight. If it is not K' -slight, there is a K' -definable infinite set X such that $X \cap (f')^{-1}(X)$ is finite. As $\text{tp}(K'/Kf')$ is finitely satisfiable in K , and infinity is definable, we can pull the parameters of X into K , finding a K -definable infinite set X_0 such that $X_0 \cap (f')^{-1}(X_0)$ is finite. This contradicts K -slightness of f' .

Next suppose X is K -displaced by f . Then X is K -displaced by f' . If X is not K' -displaced by f' , there is some $a \in X(K')$ such that $f'(a) \in X$. As $\text{tp}(a/Kf')$ is finitely satisfiable in K , there is some $a_0 \in X(K)$ such that $f'(a_0) \in X$, contradicting the fact that X is K -displaced by f' . \square

Lemma 3.6. *No K -slight map K -displaces an infinite K -definable set.*

Proof. Suppose f_0 is a K -slight map which K -displaces an infinite K -definable set Y . Inductively build a sequence of models $K_0 = K \preceq K_1 \preceq K_2 \preceq \dots$ and bijections f_0, f_1, f_2, \dots such that

- $\text{tp}(f_i/K_i)$ is an heir of $\text{tp}(f_0/K)$.
- f_i is K_{i+1} -definable.

By Lemma 3.5, f_i is K_i -slight, and K_i -displaces Y .

For $w \in \{0, 1\}^{<\omega}$, consider the set

$$Y_w = \left\{ y \in Y : \bigwedge_{i < |w|} f_i(y) \in^{w(i)} Y \right\}$$

where \in^0 denotes \notin and \in^1 denotes \in .

We will prove by induction on $|w|$ that Y_w is infinite. If we write f_i as f_{a_i} , this shows that the formula $f_x(y) \in Y$ has the independence property, a contradiction.

For the base case, Y_\emptyset is Y which is infinite by assumption.

Now suppose that Y_w is infinite; we will show Y_{w0} and Y_{w1} are infinite. Let $n = |w|$. Then Y_w is K_n -definable. If $a \in Y_w(K_n) \subset Y(K_n)$, then $f_n(a) \notin Y$ because Y is K_n -displaced by f_n . This shows that the infinite set $Y_w(K_n)$ is contained in Y_{w0} .

Also, as f_n is K_n -slight and Y_w is infinite and K_n -definable, $Y_w \cap f_n^{-1}(Y_w)$ is infinite. This set is contained in $Y_w \cap f_n^{-1}(Y) = Y_{w1}$, so Y_{w1} is infinite.

So Y_w being infinite implies Y_{w0} and Y_{w1} are infinite. This ensures that all Y_w are infinite, hence non-empty, contradicting NIP. \square

Proposition 3.7.

1. If f is a K -slight bijection and X is K -definable, then for all but finitely many $x \in K$, we have $x \in X \iff f(x) \in X$.
2. The K -slight bijections form a group under composition.

Proof. 1. Let $S \subset K$ be the externally definable set of x such that $x \in X$ and $f(x) \notin X$. We claim that S is finite. Otherwise, by Corollary 2.3, there is some infinite K -definable set Y such that $Y(K) \subset S$. Then $X \cap Y$ is an infinite K -definable set which is K -displaced by f , by choice of S . This contradicts Lemma 3.6.

So S is finite. This means that for almost all $x \in K$, we have $x \in X \implies f(x) \in X$. Replacing X with its complement, we obtain the reverse implication (with at most finitely many exceptions).

2. Suppose f and $g \circ f$ are K -slight. We will show that g is K -slight. Let X be an infinite K -definable set. Then for almost all $x \in K$, we have

$$f(x) \in X \iff x \in X \iff g(f(x)) \in X$$

So the infinite set $f(X(K))$ is almost entirely contained in $X \cap g^{-1}(X)$. Thus $X \cap g^{-1}(X)$ is infinite, for arbitrary infinite K -definable sets X . \square

Corollary 3.8. *The set I_K of K -infinitesimals is a subgroup of $(\mathbb{C}, +)$. The set I_K satisfies conditions 1-4 of §2.2. There is a unique group topology on $(K, +)$ such that $\{X -_\infty X : X \text{ is infinite and } K\text{-definable}\}$ is a neighborhood basis of 0.*

We will call this topology the *canonical topology* on K . We may also talk about the canonical topology on \mathbb{C} , because \mathbb{C} is a model just like K .

4 Germs at 0

Say that two definable sets $X, Y \subset \mathbb{C}$ have the same *germ at 0* if $0 \notin \overline{X \Delta Y}$. This is an equivalence relation. The main goal of this section is Theorem 4.7, asserting that there are only a small number of germs at 0—or equivalently, that there are only a small number of infinitesimal types over \mathbb{C} . This turns out to be the key to proving a number of basic facts about the canonical topology, as we will see in §4.1:

- Definable subsets of \mathbb{C} have finite boundary.
- Products of infinitesimals are infinitesimal.
- Products of non-infinitesimals are non-infinitesimal.
- The canonical topology has a definable basis.

To prove Theorem 4.7, we would like to mimic Pierre Simon's argument in the case of ordered dp-minimal structures (Lemma 2.10 in [14]). Matters are complicated by our lack of a definable neighborhood basis.

In what follows, we'll refer to sets of the form $X -_\infty X$ with X infinite and definable, as "basic neighborhoods (of 0)".

Let \mathcal{U} be a 0-definable family of basic neighborhoods (of 0).

Definition 4.1. *Say that \mathcal{U} is good if for every finite set $S \subset \mathbb{C}^\times$, there is some $U \in \mathcal{U}$ such that $U \cap S = \emptyset$.*

Definition 4.2. *Say that \mathcal{U} is mediocre if for every finite set $\{a_1, \dots, a_n\} \subset \mathbb{C}^\times$ of full dp-rank (of dp-rank n), there is some $U \in \mathcal{U}$ such that $U \cap S = \emptyset$.*

A good family would be helpful, but with work, a mediocre family will suffice. This is good, because of the following proposition:

Proposition 4.3. *There is a mediocre family of basic neighborhoods.*

Proof. Let $\Sigma(x)$ be the partial type over \mathbb{C} saying that $x \neq 0$ and x is a \mathbb{C} -infinitesimal.

First suppose that $\Sigma(x)$ is not finitely satisfiable in some small model K . Then there is some \mathbb{C} -definable neighborhood $U = U_b$ such that $U_b \cap K = \{0\}$. Then for all n , we have

$$\begin{aligned} \forall a_1, \dots, a_n \in K^\times : U_b \cap \{a_1, \dots, a_n\} &= \emptyset \\ \forall a_1, \dots, a_n \in K^\times \exists b \in \mathbb{C} : U_b \cap \{a_1, \dots, a_n\} &= \emptyset \\ \forall a_1, \dots, a_n \in K^\times \exists b \in K : U_b \cap \{a_1, \dots, a_n\} &= \emptyset \\ \forall a_1, \dots, a_n \in \mathbb{C}^\times \exists b \in \mathbb{C} : U_b \cap \{a_1, \dots, a_n\} &= \emptyset \end{aligned}$$

Consequently the family $\{U_b : b \in \mathbb{C}\}$ is a good family of basic neighborhoods, hence a mediocre family.

Therefore, we may assume that $\Sigma(x)$ is finitely satisfiable in any small model K . This has the following counterintuitive corollary:

Claim 4.4. *The canonical topology on K is the induced subspace topology from the canonical topology on \mathbb{C} .*

Proof. The induced subspace topology on K will have as neighborhood basis of 0, the sets of the form $N \cap K$ for N a \mathbb{C} -definable basic neighborhood. This already includes the K -definable basic neighborhoods on K , so it remains to show that if N is a \mathbb{C} -definable basic neighborhood, then there is a K -definable basic neighborhood N' such that $N' \cap K \subset N \cap K$.

By Corollary 3.8, applied to the basic neighborhoods on \mathbb{C} , there must be some \mathbb{C} -definable basic neighborhood U such that $U - U \subseteq N$. We claim $U \cap K$ is infinite.

Otherwise, by Hausdorffness we could find a smaller \mathbb{C} -definable neighborhood V such that $V \cap K = \{0\}$. This contradicts the finite satisfiability of $\Sigma(x)$ in K .

Because $U \cap K$ is infinite, it contains $Q(K)$ for some infinite K -definable set Q , by Corollary 2.3. Now $Q -_\infty Q$ is a K -definable basic neighborhood, and

$$(Q -_\infty Q) \cap K = Q(K) -_\infty Q(K) \subseteq Q(K) - Q(K) \subseteq U - U \subseteq N$$

so that $(Q -_\infty Q) \cap K \subseteq N \cap K$. Then $N' := Q -_\infty Q$ is our desired K -definable basic neighborhood. This proves the claim. \square

Claim 4.5. *There is a \emptyset -definable family of basic neighborhoods U_b such that if $K \preceq K'$ is any inclusion of models, and $a \in K' \setminus K$, then $(a + U_b) \cap K = \emptyset$ for some $b \in K'$.*

Proof. If not, then by compactness, we would obtain a pair of models $K \preceq K'$ and an element a such that every K' -definable neighborhood of a intersects K . In other words, a is in the topological closure \overline{K} of K . Embed K' into \mathbb{C} . Then K' has the induced subspace topology, so $a \in \overline{K}$ even within \mathbb{C} . Because the topology on \mathbb{C} is $\text{Aut}(\mathbb{C}/K)$ -invariant, all the conjugates of a over K are in \overline{K} , so \overline{K} is big. But in a Hausdorff topology, the closure of a set is bounded in terms of the size of the set (because every point in the closure can be written as an ultralimit of an ultrafilter on the set, and there are only a bounded number of ultrafilters). \square

Let U_b be the family from Claim 4.5. We claim that U_b is mediocre. To see this, suppose a_1, \dots, a_n are elements of \mathbb{C}^\times with dp-rank n over the empty set. By properties of dp-rank, we can find an element $t \in \mathbb{C}$ such that (\vec{a}, t) has dp-rank $n + 1$.

By subadditivity of dp-rank,

$$\begin{aligned} n + 1 &= \text{dp-rk}(t, t + a_1, \dots, t + a_n) \\ &\leq \text{dp-rk}(t/t + a_1, \dots, t + a_n) + \text{dp-rk}(t + a_1, \dots, t + a_n) \\ &\leq 1 + n \end{aligned}$$

so equality holds, and $t \notin \text{acl}(t - a_1, \dots, t - a_n)$. Therefore we can find a small model K such that $t \notin K \supseteq \{t + a_1, \dots, t + a_n\}$. By the claim there is some $b \in \mathbb{C}$ such that

$$(t + U_b) \cap \{t + a_1, \dots, t + a_n\} \subseteq (t + U_b) \cap K = \emptyset$$

so that $U_b \cap \{a_1, \dots, a_n\} = \emptyset$. \square

Lemma 4.6. *Let \mathcal{U} be a mediocre family of basic neighborhoods. Then given any small collection \mathcal{C} of infinite definable sets, there is some $U \in \mathcal{U}$ such that $C \setminus U$ is infinite for every $C \in \mathcal{C}$.*

Proof. Because infinity is definable and \mathcal{U} is a single definable family, it suffices by compactness to consider the case when \mathcal{C} is a finite collection $\{C_1, \dots, C_n\}$. By definability of infinity, there is some N (depending on \mathcal{C}) such that $C_i \setminus U$ will be infinite as long as it has size at least N .

Let A be a set over which C_1, \dots, C_n are all defined. The set $\prod_{i=1}^n C_i^N$ has dp-rank $N \cdot n$, so we can find some tuple in it, having dp-rank $N \cdot n$ over A , hence over \emptyset . By mediocrity, we can find some $U \in \mathcal{U}$ that U avoids this entire tuple. By choice of N , now each $C_i \setminus U$ is infinite. \square

Theorem 4.7. *There are only a bounded number of germs at 0 among definable subsets of \mathbb{C} .*

Proof. Suppose not.

Claim 4.8. *There is some sequence X_1, X_2, \dots of definable subsets of \mathbb{C}^\times , all belonging to a single definable family, such that $0 \in \overline{X_i}$ and $0 \notin \overline{X_i \cap X_j}$ for $i \neq j$.*

Proof. By Morley-Erdos-Rado, we can produce an indiscernible sequence of sets $Y_1, Y_2, Y_3, \dots \subseteq \mathbb{C}$ having pairwise distinct germs at 0. Let $X_i = Y_{2i} \Delta Y_{2i+1}$; then $0 \in \overline{X_i}$. By indiscernibility, 0 is in every Y_i or in none; either way each $X_i \subseteq \mathbb{C}^\times$.

By NIP, the collection $\{X_i\}$ is k -inconsistent for some k . Replace X_i with $X_{2i} \cap X_{2i+1}$ until $0 \notin \overline{X_1 \cap X_2}$. This process must terminate within $\log_2 k$ steps or so. \square

Fix X_1, X_2, \dots from the claim. Let K_1 be a small model over which the X_i are defined. Let \mathcal{U} be a mediocre famiy from Proposition 4.3. Inductively build a sequence $K_1 \preceq K_2 \preceq \dots$ and $U_1, U_2, \dots \in \mathcal{U}$ as follows:

- U_i is chosen so that $C \setminus U_i$ is infinite for every infinite K_i -definable set $C \subseteq \mathbb{C}$. This is possible by Lemma 4.6.
- K_{i+1} is chosen so that U_i is K_{i+1} -definable.

Claim 4.9. *For any i_0, j_0 , there is some a such that $a \in X_i \iff i = i_0$, and $a \in U_j \iff j < j_0$.*

Proof. By compactness, it suffices to only consider X_1, \dots, X_n and U_1, \dots, U_n . Let

$$D = X_1^c \cap X_2^c \cap \dots \cap X_{i_0-1}^c \cap X_{i_0} \cap X_{i_0+1}^c \cap \dots \cap X_n^c$$

where S^c denotes the complement $\mathbb{C} \setminus S$ of a set S .

The set D is K -definable, and $0 \in \overline{D} \setminus D$, by choice of the X_i 's. So the set

$$S = D \cap U_1 \cap \dots \cap U_{j_0-1}$$

is infinite, as $U_1 \cap \dots \cap U_{j_0-1}$ is a neighborhood of 0.

As S is K_{j_0} definable, it follows that $S \cap U_{j_0}^c$ is infinite, by choice of U_{j_0} . As $S \cap U_{j_0}^c$ is K_{j_0+1} -definable, it follows that $S \cap U_{j_0}^c \cap U_{j_0+1}^c$ is infinite. Continuing on in this fashion, we ultimately see that

$$S \cap U_{j_0}^c \cap \dots \cap U_n^c$$

is infinite. If a is any element of this set, then $a \in D$, so $a \in X_i \iff i = i_0$ (for $1 \leq i \leq n$), and

$$a \in U_1 \cap \dots \cap U_{j_0-1} \cap U_{j_0}^c \cap \dots \cap U_n^c,$$

so $a \in U_j \iff j < j_0$ (for $1 \leq j \leq n$).

Finally, using compactness, we can send n to ∞ . \square

Given the claim, the sets $\{X_i\}$ and $\{U_i \setminus U_{i+1}\}$ now directly contradict dp-minimality. \square

Corollary 4.10. *There are only a bounded number of infinitesimal types over \mathbb{C} .*

4.1 Applications of bounded germs

Using Theorem 4.7 and Corollary 4.10, we can prove a number of key facts about the canonical topology.

We will repeatedly make use of the following basic observation:

Observation 4.11. *Let $X \subset \mathbb{C}$ be K -definable, and $a \in K$. Then the following are (clearly) equivalent:*

1. *There is a K -infinitesimal ϵ such that $(a + \epsilon \in X \nleftrightarrow a \in X)$.*
2. *The type $\Sigma(x)$ asserting that $x \in I_K$ and $(a + x \in X \nleftrightarrow a \in X)$ is consistent.*
3. *For every K -definable basic neighborhood U , the set $a + U$ intersects both X and $X^c := \mathbb{C} \setminus X$.*
4. *a is in the topological boundary of $X(K)$ within K .*

Note that the third of these conditions does not depend on K , in the sense that its truth is unchanged if we replace K with an elementary extension $K' \succeq K$.

First we show that definable sets have finite boundaries.

Proposition 4.12. *If $X \subset K$ is definable, then ∂X is finite, and contained in $\text{acl}(\ulcorner X \urcorner)$.*

Proof. By Observation 4.11, we may replace K with \mathbb{C} —this only makes ∂X get bigger.

The set ∂X is type-definable, essentially by (3) of Observation 4.11. It is also type-definable over $\text{dcl}(\ulcorner X \urcorner)$, by automorphism invariance of the topology. The proposition will therefore follow if ∂X is small.

Let \mathbb{C}^* be a sufficiently saturated elementary extension of \mathbb{C} . By the equivalence of conditions 1 and 4 of Observation 4.11,

$$\partial X(\mathbb{C}) = \bigcup_{\epsilon \in I_{\mathbb{C}}} \{x \in \mathbb{C} : x + \epsilon \in X \nleftrightarrow x \in X\} \quad (1)$$

Let D_{ϵ} denote $\{x \in \mathbb{C} : x + \epsilon \in X \nleftrightarrow x \in X\}$. By the first part of Proposition 3.7, each D_{ϵ} is finite. Moreover, D_{ϵ} depends only on $\text{tp}(\epsilon/\mathbb{C})$. By Corollary 4.10, it follows that the right hand side of (1) is small. \square

Proposition 4.13. *The set I_K of K -infinitesimals is closed under multiplication. Consequently, conditions 1-5 of §2.2 hold and the canonical topology on K is a ring topology.*

Proof. Suppose ϵ and e are K -infinitesimals.

Claim 4.14. *The map $x \mapsto x \cdot (1 + e)$ is K -slight.*

Proof. Let X be an infinite K -definable set; we will show that $X \cap (1 + e)^{-1}X$ is infinite. In fact, it contains $X(K) \setminus \partial X$, which is infinite by Proposition 4.12. To see this, suppose $a \in X(K) \setminus \partial X$. Then $e \cdot a$ is K -infinitesimal by Proposition 3.2. By the equivalence of 1 and 4 in Observation 4.11 and the fact that $a \notin \partial X$, it follows that $a + e \cdot a \in X$, so $a \in X \cap (1 + e)^{-1}X$. \square

The K -slight maps are closed under composition and inverses, by Proposition 3.7. Applying this to the K -slight maps $x \mapsto (1 + e)x$ and $x \mapsto x + \epsilon$, we see that the map

$$x \mapsto (1 + e) \left(\frac{x}{1 + e} + \epsilon \right) - \epsilon = x + e \cdot \epsilon$$

is also K -slight, so $e \cdot \epsilon$ is a K -infinitesimal. \square

Lemma 4.15. *As a subgroup of the additive group, I_K has no type-definable proper subgroups of bounded index.*

Proof. By Proposition 6.1 in [5], I_K^{00} exists and is type-definable over K ; we will show $I_K^{00} = I_K$. Suppose for the sake of contradiction that there is $\epsilon \in I_K \setminus I_K^{00}$. Let K' be a model containing ϵ , and let ϵ' realize an heir of $\text{tp}(\epsilon/K)$ to K' . By Lemma 3.5, ϵ' is K' -infinitesimal.

As ϵ and ϵ' have the same (Lascar strong) type over K , they are in the same coset of I_K^{00} . Then ϵ and $\epsilon - \epsilon'$ do *not* have the same type over K , because the latter is in I_K^{00} but the former is not. Choose a K -definable set X which contains ϵ but not $\epsilon - \epsilon'$. As X is K' -definable and $\epsilon \in K'$, it follows by Observation 4.11 that $\epsilon \in \partial X$. Then by Proposition 4.12, $\epsilon \in \text{acl}(\ulcorner X \urcorner) \subseteq K$, which is absurd, since ϵ is a non-zero K -infinitesimal. \square

Theorem 4.16. *The canonical topology on K is a V -topology. The set I_K satisfies conditions 1-7 of §2.2, and is the maximal ideal of a valuation ring \mathcal{O}_K on \mathbb{C} .*

Proof. By Observation 2.4 and Proposition 4.13, it suffices to show that the complement of I_K is closed under multiplication (condition 7 of §2.2).

First we prove a general fact about dp-minimal groups.

Claim 4.17. *Suppose G and H are type-definable subgroups of $(K, +)$, such that $G = G^{00}$ and $H = H^{00}$. Then $G \subseteq H$ or $H \subseteq G$.*

Proof. Otherwise, $G \cap H$ has unbounded index in both G and H . By Morley-Erdős-Rado we can produce an indiscernible sequence $\langle (a_i, b_i) \rangle_{i < \omega + \omega}$ of elements of $G \times H$ such that the a_i are in pairwise distinct cosets of $G \cap H$, and the b_i are in pairwise distinct cosets of $G \cap H$. The sequences a_0, a_1, \dots and $b_\omega, b_{\omega+1}, \dots$ are mutually indiscernible. However, after naming $c := a_0 + b_\omega$, neither sequence is indiscernible. Indeed, $c - a_i \in H$ if and only if $i = 0$, and $b_{\omega+i} - c \in G$ if and only if $i = 0$. This contradicts the characterization of dp-rank 1 in terms of mutually indiscernible sequences. \square

Now, let R be the set of $a \in \mathbb{C}$ such that $a \cdot I_K \subseteq I_K$. Observe:

1. R is closed under multiplication, trivially.
2. R is closed under addition and subtraction, because I_K is closed under addition and subtraction. So R is a ring.
3. R is a valuation ring in \mathbb{C} : for any $a \in \mathbb{C}^\times$, the groups $a \cdot I_K$ and I_K are comparable by Claim 4.17 and Lemma 4.15. If $a \cdot I_K \subseteq I_K$, then $a \in R$, and if $I_K \subseteq a \cdot I_K$, then $1/a \in R$.

4. I_K is contained in R (by Proposition 4.13), so I_K is an ideal in R .
5. I_K is a proper ideal of R , because $1 \notin I_K$.
6. K is contained in R , by Proposition 3.2, specifically the fact that I_K is closed under multiplication by K , which is condition 4 of §2.2.

By general facts about valuation rings, the set $J_K = \{x \in \mathbb{C} : x^2 \in I_K\}$ is also a proper ideal in R , and $\mathbb{C} \setminus I_K$ is closed under multiplication if and only if $I_K = J_K$.

Because J_K is a proper ideal in a superring of K , J_K satisfies conditions 2-4 of §2.2: it is closed under multiplication by K , it is closed under addition and subtraction, and its intersection with K is $\{0\}$. As $J_K \supseteq I_K$, it also satisfies condition 1, the non-triviality condition that $J_K \supsetneq \{0\}$.

So J_K and I_K both satisfy conditions 1-4 of §2.2. Choose nonzero $a \in I_K$. Then $a \cdot J_K \subseteq I_K$ because I_K is an ideal. By Observation 2.5,

$$a \cdot J_K \subseteq I_K \subseteq J_K$$

implies $J_K \subseteq I_K \subseteq J_K$, completing the proof. \square

Corollary 4.18. *The canonical topology has a definable basis of opens. More precisely, there is a definable open set B such that sets of the form $a \cdot B$ for $a \in \mathbb{C}^\times$ form a neighborhood basis of 0, and consequently the sets of the form $a \cdot B + b$ form a basis for the topology.*

Proof. Let \mathcal{O} be the valuation ring whose maximal ideal is I_K . For any $x \in \mathbb{C}^\times$,

$$x \in \mathcal{O} \iff x^{-1} \notin I_K$$

so that \mathcal{O} is \vee -definable, over K . By compactness, there is some K -definable set B such that

$$I_K \subseteq B \subseteq \mathcal{O}$$

The inclusion $I_K \subseteq B$ means that B is a neighborhood of 0. Replacing B with B^{int} (which is still K -definable, by Proposition 4.12), we may assume that B is open.

We claim that $\{a \cdot B : a \in K^\times\}$ is a neighborhood basis of 0 in the canonical topology on K . Let U be a K -definable neighborhood of 0. Take ϵ a non-zero K -infinitesimal. Then

$$\epsilon \cdot B \subseteq \epsilon \cdot \mathcal{O} \subseteq I_K \subseteq U$$

As $K \preceq \mathbb{C}$, there is some $a \in K^\times$ such that $a \cdot B \subseteq U$. This shows that $\{a \cdot B : a \in K^\times\}$ is a neighborhood basis of 0. Throwing in translates, we get a basis for the topology. \square

Definition 4.19. *A standard ball is an open definable set $B \subset \mathbb{C}$ such that $\{a \cdot B : a \in \mathbb{C}^\times\}$ is a neighborhood basis of 0.*

Remark 4.20. *Suppose B is a K -definable standard ball and ϵ is a K -infinitesimal. Then $\epsilon \cdot B \subseteq I_K$ and hence $\epsilon \cdot B$ is contained in any K -definable neighborhood of 0.*

Proof. Let $v(-)$ be the valuation on \mathbb{C} whose maximal ideal is I_K . As in the proof of Corollary 4.18, there is some K -definable set B' containing the maximal ideal I_K and contained in the valuation ring. In particular, B' is a neighborhood of 0 and elements of B' have nonnegative valuation. Because B is a standard ball, there is some $a \in \mathbb{C}^\times$ such that $a \cdot B \subseteq B'$. As B and B' are K -definable, we can take $a \in K$. Neither a nor a^{-1} is K -infinitesimal, so $v(a) = 0$, and we see that $v(x) \geq 0$ for any $x \in B$.

Now if ϵ is a K -infinitesimal, then every element of $\epsilon \cdot B$ has positive valuation, so $\epsilon \cdot B \subseteq I_K$. \square

5 Henselianity

Let \mathcal{O}_K be the valuation ring whose maximal ideal is I_K . (This notation is a bit unfortunate, since \mathcal{O}_K is a valuation ring on \mathbb{C} , not K . In fact, the valuation is trivial on K .)

In this section, we prove that \mathcal{O}_K is a henselian valuation ring.

5.1 Finding interior

Lemma 5.1. *Naming infinitesimals does not algebraize anything, in the following sense:*

1. Let $\mathbb{C}^* \succeq \mathbb{C}$ be an elementary extension, and $\epsilon \in \mathbb{C}^*$ be \mathbb{C} -infinitesimal. For any small $S \subset \mathbb{C}$, we have $\mathbb{C} \cap \text{acl}(S\epsilon) = \text{acl}(S)$.
2. Let p be an infinitesimal type over \mathbb{C} . Suppose $S \subset \mathbb{C}$ is small, $a \in \mathbb{C}$, and $\epsilon \models p|Sa$. Then $a \in \text{acl}(S) \iff a \in \text{acl}(S\epsilon)$.

Proof. 1. Fix S . For $\epsilon \in I_{\mathbb{C}}$, let $X_\epsilon = \text{acl}(S\epsilon) \cap \mathbb{C}$. Then X_ϵ is small and depends only on $\text{tp}(\epsilon/\mathbb{C})$. By Corollary 4.10, it follows that $\bigcup_{\epsilon \in I_{\mathbb{C}}} X_\epsilon$ is small. It is also $\text{Aut}(\mathbb{C}/S)$ -invariant, so it must be contained in $\text{acl}(S)$. In particular, $X_\epsilon \subseteq \text{acl}(S)$ for any \mathbb{C} -infinitesimal ϵ .

2. Let $\mathbb{C}^* \succeq \mathbb{C}$ be an elementary extension in which p is realized by some ϵ' . Then $\epsilon' \equiv_{aS} \epsilon$, so

$$a \in \text{acl}(S\epsilon) \iff a \in \text{acl}(S\epsilon') \iff a \in \text{acl}(S)$$

where the second equivalence follows by the previous point. \square

If S is a small set, say that an n -tuple (a_1, \dots, a_n) is *algebraically independent over S* if $a_i \notin \text{acl}(S, a_{\neq i})$ for each i .

Lemma 5.2. *Let S be a small set over which some standard ball B is defined. Let (a_1, \dots, a_n) be algebraically independent over S . If \vec{a} is in an S -definable set $Y \subset \mathbb{C}^n$, then $\vec{a} \in Y^{\text{int}}$ (in the product topology on \mathbb{C}^n).*

Proof. We proceed by induction on n . The case $n = 1$ is Proposition 4.12. Suppose $n > 1$. Let K be a small model containing S, a_1, \dots, a_n . Let p be some global infinitesimal type and let ϵ realize $p|K$.

Let Y be the set of $x_1 \in \mathbb{C}$ such that $(x_1, a_2, \dots, a_n) \in X$. Then Y is $Sa_2 \cdots a_n$ -definable, so $a_1 \notin \partial Y$ by Proposition 4.12. Then $Y - a_1$ is a K -definable neighborhood of 0, so it contains $\epsilon \cdot B$ by Remark 4.20. Thus $a_1 + \epsilon \cdot B \subseteq Y$.

Consider

$$Z = \{(x_2, \dots, x_n) \in \mathbb{C}^{n-1} : (a_1 + \epsilon \cdot B) \times \{(x_1, \dots, x_n)\} \subseteq X\}$$

This set is $Sa_1\epsilon$ -definable, and contains (a_2, \dots, a_n) . By Lemma 5.1, (a_2, \dots, a_n) is algebraically independent over $Sa_1\epsilon$, so by induction, (a_2, \dots, a_n) is in the interior of Z . If U is any neighborhood of (a_2, \dots, a_n) in Z , then $(a_1 + \epsilon \cdot B) \times U$ is a neighborhood of (a_1, \dots, a_n) in X . \square

Proposition 5.3. *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a finite-to-one definable map. If $X \subset \mathbb{C}^n$ is a set with non-empty interior, then $f(X)$ also has non-empty interior. (We are not assuming X is definable.)*

Proof. The topology on \mathbb{C}^n has a definable basis, so X must contain a definable open. Shrinking X , we may assume X is definable. Choose a small model K such that X and f are K -definable and some standard ball is K -definable. As X has interior, it has dp-rank n . Choose $\vec{a} \in X$ with $\text{dp-rk}(\vec{a}/K) = n$ and let $\vec{b} = f(\vec{a})$. The tuple \vec{b} is interalgebraic with \vec{a} over K , so $\text{dp-rk}(\vec{b}/K) = n$. By dp-minimality and subadditivity of dp-rank, this implies \vec{b} is algebraically independent over K (otherwise, \vec{b} could have dp-rank at most $n - 1$). By Lemma 5.2, \vec{b} is in the interior of $f(X)$. \square

5.2 Henselianity

Lemma 5.4. *Let F be a field with some structure, and L/F be a finite extension. Suppose \mathcal{O} is a \vee -definable valuation ring on F . Then each extension of \mathcal{O} to L is \vee -definable (over the same parameters used to define \mathcal{O} and interpret L).*

Proof. Replacing L with the normal closure of L over F , we may assume L/F is a normal extension of some degree n .

Claim 5.5. *There is some $d = d(k, n)$ such that the following are equivalent for $\{a_1, \dots, a_k\} \subset L$:*

- No extension of \mathcal{O} to L contains $\{a_1, \dots, a_k\}$.
- $1 = P(a_1, \dots, a_k)$ for some polynomial $P(X_1, \dots, X_k) \in \mathfrak{m}[X_1, \dots, X_k]$ of degree less than $d(k, n)$.

Proof. Consider the theory T_n whose models consist of degree n normal field extensions L/F with a predicate picking out a valuation ring \mathcal{O}_L on L . On general valuation-theoretic grounds, the following are equivalent for $\{a_1, \dots, a_k\} \subset L$

- $\{a_1, \dots, a_k\} \not\subseteq \sigma(\mathcal{O}_L)$ for any $\sigma \in \text{Aut}(L/F)$.
- No extension of $\mathcal{O}_L \cap F$ to L contains $\{a_1, \dots, a_k\}$.
- $1 = P(a_1, \dots, a_k)$ for some $P(X_1, \dots, X_k) \in \mathfrak{m}[X_1, \dots, X_k]$.

The first condition is a definable condition on the k -tuple (a_1, \dots, a_k) , so by compactness applied to T_n , there is a bound on the degree in the third condition. \square

Because \mathcal{O} is \vee -definable, \mathfrak{m} is type-definable, so the second condition in the claim is type-definable.

Let \mathcal{O}' be some extension of \mathcal{O} to L . We can find some finite set $S \subseteq \mathcal{O}'$ such that \mathcal{O}' is the unique extension of \mathcal{O} containing S , because there are only finitely many extensions and they are pairwise incomparable. The claim implies type-definability of the set

$$\{x \in L : \text{no extension of } \mathcal{O} \text{ to } L \text{ contains } S \cup \{x\}\}$$

which is the complement of \mathcal{O}' by choice of S . \square

Recall that \mathcal{O}_K denotes the valuation ring whose maximal ideal is I_K .

Proposition 5.6. *Let K be a small submodel of \mathbb{C} . Let L/K be a finite algebraic extension, and $\mathbb{L} = L \otimes_K \mathbb{C}$. (So \mathbb{L} is a saturated elementary extension of L .) Then \mathcal{O}_K has a unique extension to \mathbb{L} .*

Proof. Let $\mathcal{O}_1, \dots, \mathcal{O}_m$ denote the extensions of \mathcal{O}_K to \mathbb{L} . By Lemma 5.4, these are all \vee -definable over K . Let \mathfrak{m}_i be the maximal ideal of \mathcal{O}_i ; this is type-definable over K . Let v_i be the valuation on \mathcal{O}_i .

Each v_i is a non-trivial valuation, which is trivial when restricted to K or even L . It follows easily that each \mathfrak{m}_i satisfies conditions 1-7 of §2.2.

Let I_L denote $\bigcap_i \mathfrak{m}_i$. Then I_L satisfies conditions 2-6 of §2.2 because the \mathfrak{m}_i do, and it satisfies condition 1 (non-triviality) because it contains I_K .

So the \mathcal{O}_i determine V -topologies on L , and I_L determines a field topology on L .

Claim 5.7. *The topology on I_L is the product topology on L (thinking of L as a finite-dimensional K -vector space.)*

Proof. Write $L = K(\alpha)$ (possible by Observation 2.1). So $\mathbb{L} = \mathbb{C}(\alpha)$, and $\{1, \alpha, \dots, \alpha^{n-1}\}$ is a basis for \mathbb{L} over \mathbb{C} .

Let (F, \mathcal{O}) be some algebraically closed valued field extending $(\mathbb{C}, \mathcal{O}_K)$. Let ι_1, \dots, ι_n denote the embeddings of \mathbb{L} into F . Then

$$\begin{aligned} \{\mathcal{O}_1, \dots, \mathcal{O}_m\} &= \{\iota_i^{-1}(\mathcal{O}) : 1 \leq i \leq n\} \\ \{\mathfrak{m}_1, \dots, \mathfrak{m}_m\} &= \{\iota_i^{-1}(\mathfrak{m}) : 1 \leq i \leq n\} \end{aligned}$$

where \mathfrak{m} is the maximal ideal of \mathcal{O} . (This is merely saying that all the extensions of \mathcal{O}_K to \mathbb{L} are obtained by embeddings of \mathbb{L} into F .)

Because $K \subseteq \mathcal{O}$ (as no element of K is the reciprocal of a K -infinitesimal), it follows that $K^{alg} \subseteq \mathcal{O}$, where K^{alg} is the algebraic closure of K inside F . Let $\alpha_1, \dots, \alpha_n$ be the images of α under ι_1, \dots, ι_n . These are pairwise distinct because \mathbb{L}/\mathbb{C} is separable (by Observation 2.1). Let M be the Vandermonde matrix whose (i, j) entry is α_i^{j-1} . Then $M \in GL_n(K^{alg}) \subseteq GL_n(\mathcal{O})$.

It follows that multiplication by M and M^{-1} preserves $\mathcal{O}^n \subseteq F^n$, as well as $\mathfrak{m}^n \subseteq F^n$. Concretely, this means that if $(x_0, x_1, \dots, x_{n-1}) \in F^n$, then the following are equivalent:

- Each $x_i \in \mathfrak{m}$
- $\sum_{i=0}^{n-1} x_i \alpha_j^i \in \mathfrak{m}$ for each j .

Specializing to the case where $x_0, \dots, x_{n-1} \in \mathbb{C}$, and writing $x = \sum_{i=0}^{n-1} x_i \alpha^i$, the following are equivalent:

- The coordinates of x (with respect to the basis $\{1, \dots, \alpha^{n-1}\}$) are in I_K
- $\iota_j(x) \in \mathfrak{m}$ for each $j \leq n$, or equivalently, $x \in \mathfrak{m}_i$ for each $i \leq m$.

The latter of these means that $x \in I_L$, so we see that

$$I_L = I_K + I_K \cdot \alpha + \dots + I_K \cdot \alpha^{n-1}$$

from which it is clear that I_L corresponds to the product topology. \square

Our goal is to prove that $\mathcal{O}_1, \dots, \mathcal{O}_m$ are all equal (i.e., $m = 1$). Suppose otherwise.

First suppose that K does not have characteristic 2. By some fact related to Stone approximation (see the proof of (4.1) in [12]), we can find an element x such that $x \in 1 + \mathfrak{m}_1$ and $x \in -1 + \mathfrak{m}_i$ for $i > 1$. Note that $x^2 \in 1 + \mathfrak{m}_i$ for all i .

Thus $x \notin 1 + I_L$, $x \notin -1 + I_L$, and $x^2 \in 1 + I_L$.

Because I_L defines a field topology, $1 + I_L$ is a subgroup of \mathbb{L}^\times . It is also topologically open: if B is a standard ball and ϵ is K -infinitesimal, then $\epsilon \cdot B \subseteq I_K$ (by Remark 4.20), so by the Claim

$$1 + \epsilon \cdot B + \epsilon \cdot B \cdot \alpha + \dots + \epsilon \cdot B \cdot \alpha^{n-1}$$

is an open set inside $1 + I_L$.

The squaring map on \mathbb{L}^\times is finite-to-one, so by Proposition 5.3, $(1 + I_L)^2$ has interior. Since $(1 + I_L)^2$ is a group, it is actually open, hence contains a neighborhood of 1:

$$(1 + I_L)^2 \text{ is a neighborhood of } 1 \tag{2}$$

Now $x \notin 1 + I_L$ and $-x \notin 1 + I_L$, and I_L is type-definable over K . So there is some K -definable set U containing I_L , such that $x \notin 1 + U$ and $-x \notin 1 + U$. By (2), $(1 + U)^2$ is a neighborhood of 0. It is K -definable, so it contains $1 + I_L$, hence x^2 . Then there is $y \in 1 + U$ such that $y^2 = x^2$. Either $x \in 1 + U$ or $-x \in 1 + U$, contradicting the choice of U .

If K has characteristic 2, replace -1 and 1 with 0 and 1 , replace the squaring map with the Artin-Schreier map, and replace $1 + I_L < \mathbb{L}^\times$ with $I_L < \mathbb{L}$. \square

Lemma 5.8. *If \mathcal{O} is a non-trivial definable valuation ring on \mathbb{C} , then \mathcal{O} induces the canonical topology on \mathbb{C} . If \mathcal{O} is K -definable, then \mathcal{O}_K is a coarsening of \mathcal{O} .*

Proof. It suffices to show that $I_K \subseteq \mathcal{O} \subseteq \mathcal{O}_K$, which implies both that \mathcal{O}_K is a coarsening of \mathcal{O} and that \mathcal{O} is a standard ball as in the proof of Corollary 4.18.

Let \mathfrak{m} be the maximal ideal of \mathcal{O} . It is infinite, since \mathcal{O} is non-trivial. By Proposition 4.12, \mathfrak{m} has interior. As \mathfrak{m} is a subgroup of the additive group, \mathfrak{m} is open. Then 0 is in the interior of \mathfrak{m} , meaning that $I_K \subseteq \mathfrak{m}$. This directly implies $\mathcal{O} \subseteq \mathcal{O}_K$. \square

Remark 5.9. *Suppose F is a field with some structure, and \mathcal{O}_1 and \mathcal{O}_2 are incompatible \vee -definable valuation rings on F . Then the join $\mathcal{O}_1 \mathcal{O}_2$ is definable.*

Proof. The join can be written as either $\{x \cdot y : x \in \mathcal{O}_1, y \in \mathcal{O}_2\}$ (which is \vee -definable) or as $\{x \cdot y : x \in \mathfrak{m}_1, y \in \mathfrak{m}_2\}$, which is type-definable.¹ \square

Lemma 5.10. *Let \mathbb{L}/\mathbb{C} be a finite algebraic extension. Any two non-trivial definable valuation rings on \mathbb{L} are not independent, i.e., they induce the same topology.*

Proof. Let w_1, w_2 be two definable valuations on \mathbb{L} , and let v_1 and v_2 be their restrictions to \mathbb{C} . Let Γ_i be the value group of w_i . Let K be a small model over which everything is defined (including the extension \mathbb{L}/\mathbb{C}). Let v_K be the non-definable valuation on \mathbb{C} coming from \mathcal{O}_K and I_K . By Lemma 5.8, v_K is a coarsening of v_1 and v_2 . So there are convex subgroups $\Delta_i < \Gamma_i$ such that v_K is equivalent to the coarsening of v_i by Δ_i . Let w'_i be the coarsening of w_i by Δ_i . Then w'_1 and w'_2 are valuations on \mathbb{L} extending v_K . By Proposition 5.6, w'_1 and w'_2 are equivalent (because v_K has an essentially unique extension). It follows that w_1 and w_2 have a common coarsening—the unique extension of v_K to \mathbb{L} . This common coarsening is non-trivial, because v_K is non-trivial. Non-trivial coarsenings induce the same topology, so w_1, w'_1 , and w_2 all induce the same topology. Therefore w_1 and w_2 are not independent. \square

Proposition 5.11. *Let \mathbb{L} be a finite extension of \mathbb{C} . Any two definable valuation rings on \mathbb{L} are comparable.*

Proof. Suppose \mathcal{O}_1 and \mathcal{O}_2 are incomparable. Let $\mathcal{O} = \mathcal{O}_1 \cdot \mathcal{O}_2$ be their join, which is definable by Remark 5.9. Let w be the valuation corresponding to \mathcal{O} , and let v be its restriction to \mathbb{C} .

The residue field $\mathbb{L}' := \mathbb{L}w$ is a finite extension of $\mathbb{C}' := \mathbb{C}v$. Moreover, \mathbb{L}' has two independent definable valuations, induced by \mathcal{O}_1 and \mathcal{O}_2 . This ensures that \mathbb{L}' is infinite and unstable, so \mathbb{C}' is also infinite and unstable. But \mathbb{C}' has dp-rank at most 1, so \mathbb{C}' is a dp-minimal unstable field. It is also as saturated as \mathbb{C} , so all our results so far apply to \mathbb{C}' . By Lemma 5.10, \mathbb{L}' cannot have two independent definable valuation rings, a contradiction. \square

Corollary 5.12. *Any definable valuation ring \mathcal{O} on \mathbb{C} is henselian.*

Proof. Otherwise, \mathcal{O} would have two incomparable extensions to some finite Galois extension of \mathbb{C} . \square

Corollary 5.12 was obtained independently by Jahnke, Simon, and Walsberg (Proposition 4.5 in [7]).

Theorem 5.13. *The valuation ring \mathcal{O}_K (whose maximal ideal is the set of K -infinitesimals) is henselian.*

Proof. Suppose not. Then \mathcal{O}_K has multiple extensions to some finite algebraic extension \mathbb{L}/\mathbb{C} . Let \mathcal{O}_1 and \mathcal{O}_2 be two such extensions. Let $K' \succeq K$ be a larger model over which the field extension \mathbb{L}/\mathbb{C} is defined. As $I_{K'} \subseteq I_K$, we see that $\mathcal{O}_{K'}$ is a coarsening

¹Here, we are using the fact that if \mathcal{O} is a valuation ring with maximal ideal \mathfrak{m} , and S is any set, then $S \cdot \mathcal{O}$ and $S \cdot \mathfrak{m}$ are closed under addition, and are equal to each other unless S has an element of minimum valuation. Incomparability of \mathcal{O}_1 and \mathcal{O}_2 ensures that e.g. $v_2(\mathcal{O}_2)$ has no minimum.

of \mathcal{O}_K . Also, $\mathcal{O}_{K'}$ has a unique extension to \mathbb{L} by Proposition 5.6. As in the proof of Lemma 5.10, this ensures that \mathcal{O}_1 and \mathcal{O}_2 are not independent. Their join $\mathcal{O}_1 \cdot \mathcal{O}_2$ is definable by Lemma 5.4 and Remark 5.9. It is also non-trivial because \mathcal{O}_1 and \mathcal{O}_2 aren't independent.

So there is some definable non-trivial valuation ring on \mathbb{C} . The property of being a valuation ring is expressed by finitely many sentences, and $K \preceq \mathbb{C}$, so there is a K -definable non-trivial valuation ring \mathcal{O} . This ring is henselian by Corollary 5.12, and \mathcal{O}_K is a coarsening, by Lemma 5.8. Coarsenings of henselian valuations are henselian. \square

5.3 Summary of results so far

In what follows, we will need only the following facts from §3-§5:

Theorem 5.14. *Let K be a dp-minimal field.*

1. K is perfect.
2. If K is sufficiently saturated and not algebraically closed, then K admits a non-trivial Henselian valuation (not necessarily definable).
3. Any definable valuation on K is henselian. Any two definable valuations on K (or any finite extension of K) are comparable.
4. For any n , the cokernel of the n th power map $K^\times \rightarrow K^\times$ is finite.

Proof.

1. Observation 2.1.
2. If K is strongly minimal, then K is algebraically closed by a well-known theorem of Macintyre. Otherwise, this is Theorem 5.13.
3. If K isn't strongly minimal, this is Proposition 5.11 and Corollary 5.12. Otherwise, K is NSOP, so has only the trivial valuation.
4. If K is strongly minimal, then K is algebraically closed (Macintyre), so the cokernels are always trivial. If $K^\times/(K^\times)^n$ is infinite, we can find some elementary extension $M \succeq K$ such that $M^\times/(M^\times)^n$ is greater in cardinality than the total number of infinitesimal types over \mathbb{C} , by Corollary 4.10. By Lemma 3.5, heirs of infinitesimal types are infinitesimal types, so \mathbb{C} has at least as many infinitesimal types as M , and therefore the cardinality of $M^\times/(M^\times)^n$ exceeds the number of infinitesimal types over M . Now for any $a \in M^\times$, and any M -infinitesimal ϵ , the element $a \cdot \epsilon^n$ is an M -infinitesimal in the same coset as a . So there are M -infinitesimals in every coset of $(M^\times)^n$, contradicting the choice of M .

\square

6 The proof of Theorem 1.2

6.1 Review of Jahnke-Koenigsmann

First we review some facts and definitions from [6].

Following [6], if K is any field, let $K(p)$ denote the compositum of all p -nilpotent Galois extensions of K . Let's say that K is “ p -closed” if $K = K(p)$. The map $K \mapsto K(p)$ is a closure operation on the subfields of K^{alg} . By an analogue of the Artin-Schreier theorem, if $[K(p) : K]$ is finite, then K is p -closed or orderable. Say that a field K is “ p -jammed” if no finite extension is p -closed.

Remark 6.1. *If K is not real closed or separably closed, then K has a finite extension which is p -jammed for some prime p .*

Proof. Replace K with $K(\sqrt{-1})$ in characteristic 0. Take some non-trivial finite Galois extension L/K . Take p dividing $|\text{Gal}(L/K)|$. By Sylow theory there is some intermediate field $K < F < L$ such that L/F is a p -nilpotent Galois extension. Then $F(p) \neq F$, so $[F(p) : F] = \infty$ because F isn't orderable. No finite extension F' of F will contain $F'(p) \supseteq F(p)$, so F is p -jammed. \square

Following [6], a valuation on a field K is p -henselian if it has a unique extension to $K(p)$. On any field K , there is a *canonical p -henselian valuation* v_K^p . If the residue field Kv_K^p is not p -closed, then v_K^p is the finest p -henselian valuation on K . By Main Theorem 3.1 of [6], v_K^p is 0-definable, provided that $X^{p^2} - 1$ splits in K , and $K(p) \neq K$.

6.2 Applying Jahnke-Koenigsmann

Theorem 6.2. *Let K be a sufficiently saturated dp-minimal field. Let \mathcal{O}_∞ be the intersection of all the definable valuation rings on K . (So $\mathcal{O}_\infty = K$ if K admits no definable non-trivial valuations.)*

1. \mathcal{O}_∞ is a henselian valuation ring on K
2. \mathcal{O}_∞ is type-definable, without parameters.
3. The residue field of \mathcal{O}_∞ is finite, real-closed, or algebraically closed. If it is finite, then \mathcal{O}_∞ is definable.

Proof. 1. By Theorem 5.14(3), the class of definable valuation rings on K is totally ordered. An intersection of a chain of valuation rings is a valuation ring. An intersection of a chain of henselian valuation rings is henselian.

2. We need to show that \mathcal{O}_∞ is a small intersection. Suppose \mathcal{O} is a definable valuation ring on K , defined by a formula $\phi(K; b)$. Let $\psi(x)$ be the formula asserting that $\phi(K; x)$ is a valuation ring. Then $\bigcap_{b \in \psi(K)} \phi(K; b)$ is a \emptyset -definable valuation ring contained in \mathcal{O} . Thus every definable valuation ring on K contains a 0-definable valuation ring. Therefore \mathcal{O}_∞ is the intersection of the 0-definable valuation rings on K . It is therefore type-definable over \emptyset .

3. Let v_∞ denote the valuation on K corresponding to \mathcal{O}_∞ . If v is a valuation on K , let Kv denote the residue field of v . If v is henselian and L/K is a finite extension, write the residue field of v 's unique extension to L as Lv , by abuse of notation.

We'll refer to a map $K \cup \{\infty\} \rightarrow k \cup \{\infty\}$ coming from a residue map, as a “place” from K to k .

Write $v \leq v'$ to indicate that v is a coarsening of v' . In this case, there are places $Kv \rightarrow Kv'$ and even $Lv \rightarrow Lv'$. If v is definable, then $v \leq v_\infty$ by choice of \mathcal{O}_∞ . If L/K is a finite extension, the definable valuations on L are totally ordered, and in strict order-preserving bijection with the definable valuations on K , essentially by Theorem 5.14.

Remark 6.3. *If v is definable and $Lv \rightarrow k$ is a definable place, then $Lv \rightarrow k$ is equivalent to $Lv \rightarrow Lv'$ for some definable $v' \geq v$ on K .*

(To see this, compose the places $L \rightarrow Lv \rightarrow k$ to get a definable valuation on L ; let v' be its restriction to K .)

With these preliminaries out of the way, we first show that Kv_∞ is perfect. Suppose Kv_∞ has characteristic p . Then Kv_1 has characteristic p for some definable valuation v_1 . (Otherwise, K has characteristic 0 and $1/p \in \mathcal{O}$ for every definable valuation ring on K . So $1/p \in \mathcal{O}_\infty$, and Kv_∞ has characteristic 0.) The field Kv_1 is finite or dp-minimal, so by Theorem 5.14, it is perfect. The place $Kv_1 \rightarrow Kv_\infty$ now ensures that Kv_∞ is perfect as well.

Now we turn to proving part 3 of the theorem. First suppose that Kv is finite for some definable v . Finite fields have only the trivial valuation, so the place $Kv \rightarrow Kv_\infty$ must be the identity map. This forces $v_\infty = v$, in which case statement 3 holds.

So, assume that Kv is infinite for all definable valuations v . We will show that Kv_∞ is real closed or algebraically closed. Suppose not. Then some finite extension of Kv_∞ is p -jammed by Remark 6.1, for some prime p . It follows that $Lv_\infty(p) \neq Lv_\infty$ for all sufficiently big finite extensions of K .

Let v_1 be a definable valuation on K such that $Kv_1 \rightarrow Kv_\infty$ is pure characteristic. (We saw that such a v_1 must exist, when proving that Kv_∞ is perfect.) Let L/K be a finite extension that is sufficiently big, so that

- Lv_1 has all the p^2 th roots of unity.
- Lv_∞ is not p -closed.

By the main theorem of [6], the canonical p -henselian valuation on Lv_1 is definable. By Remark 6.3, the canonical p -henselian place is $Lv_1 \rightarrow Lv_2$ for some $v_2 > v_1$. By definition of the canonical p -henselian place, either $Lv_1 \rightarrow Lv_2$ is the finest p -henselian place on Lv_1 , or Lv_2 is p -closed.

As Lv_1 has all the p^2 th roots of unity, the same holds for Lv_2 and Lv_∞ . Consequently, p -closedness is equivalent to surjectivity of the p th power map or surjectivity of the Artin-Schreier map (depending on the characteristic). By choice of v_1 , the fields Lv_1, Lv_2, Lv_∞ all have the same characteristic. Consequently, the place $Lv_2 \rightarrow Lv_\infty$ ensures that

$$Lv_2 = Lv_2(p) \implies Lv_\infty = Lv_\infty(p)$$

As Lv_∞ is not p -closed, neither is Lv_2 . Therefore $Lv_1 \rightarrow Lv_2$ is the finest p -henselian valuation on Lv_1 .

By assumption, Kv_2 is infinite. So it has dp-rank 1. If Kv_2 is algebraically closed, then so is Lv_2 ; but we just showed that Lv_2 is not p -closed. So Kv_2 is a dp-minimal

field which is not algebraically closed. By Theorem 5.14(2), Kv_2 and its finite extension Lv_2 admit non-trivial henselian valuations. So there is some non-trivial henselian place $Lv_2 \rightarrow k$. The place $L \rightarrow Lv_2$ is henselian by Theorem 5.14(3), so $Lv_1 \rightarrow Lv_2$ is henselian. Compositions of henselian places are henselian, so $Lv_1 \rightarrow Lv_2 \rightarrow k$ is henselian, hence p -henselian. Then $Lv_1 \rightarrow Lv_2 \rightarrow k$ is a p -henselian place that is strictly finer than $Lv_1 \rightarrow Lv_2$, contradicting the canonicity of $Lv_1 \rightarrow Lv_2$. □

6.3 Wrapping up

In what follows, we will repeatedly use the *Shelah expansion*. If M is an NIP structure, M^{sh} denotes the expansion of M by all externally definable sets. By [15] Proposition 3.23, M^{sh} eliminates quantifiers. Using this, it is easy to check that M^{sh} is dp-minimal when M is dp-minimal.

In particular, if K is our sufficiently saturated dp-minimal field, then K^{sh} is also dp-minimal (though probably no longer saturated).

Definition 6.4. A valuation $v : K \rightarrow \Gamma$ is roughly p -divisible if $[-v(p), v(p)] \subset p \cdot \Gamma$, where $[-v(p), v(p)]$ denotes $\{0\}$ in pure characteristic 0, denotes Γ in pure characteristic p , and denotes $[-v(p), v(p)]$ in mixed characteristic.

Remark 6.5. Let P be one of the following properties of valuation data:

- Roughly p -divisible
- Henselian
- Henselian and defectless
- Every countable chain of balls has non-empty intersection

If $K_1 \rightarrow K_2$ and $K_2 \rightarrow K_3$ are places, the composition $K_1 \rightarrow K_3$ has property P if and only if each of $K_1 \rightarrow K_2$ and $K_2 \rightarrow K_3$ has property P . (In each case, this is straightforward to check.)

Definition 6.6. Say that a field K has nothing to do with p if p does not divide $[L : K]$ for every finite extension L .

Remark 6.7.

1. By Corollary 4.4 of [9] and Theorem 5.14.1, any dp-minimal field of characteristic p has nothing to do with p .
2. If K has nothing to do with p , then any henselian valuation on K with residue characteristic p is defectless, and has p -divisible value group.

Lemma 6.8. Let (K, v) be a mixed characteristic henselian field, having dp-rank 1 as a valued field. Then v is defectless. If absolute ramification is unbounded, then v is roughly p -divisible, where p is the residue characteristic.

Here, “absolute ramification is unbounded” means that the interval $[-v(p), v(p)]$ is infinite in the value group.

Proof. Both conditions (defectlessness and rough p -divisibility) are first-order, so we may assume (K, v) is saturated. Let Γ be the value group. Let Δ_0 be the biggest convex subgroup not containing $v(p)$, and Δ be the smallest convex subgroup containing $v(p)$. (So, Δ_0 is non-trivial exactly if absolute ramification is unbounded.) If K_3 denotes the residue field of K , then these two convex subgroups decompose the place $K \rightarrow K_3$ as a composition of three henselian places:

$$K \xrightarrow{\Gamma/\Delta} K_1 \xrightarrow{\Delta/\Delta_0} K_2 \xrightarrow{\Delta_0} K_3 \quad (3)$$

where each arrow is labeled by its value group. The fields K and K_1 have characteristic zero, and K_2 and K_3 have characteristic p .

Both Δ_0 and Δ are externally definable, hence definable in the dp-minimal field K^{sh} . So the above sequence of places is interpretable in K^{sh} . In particular, K_2 is a dp-minimal field of characteristic p , so K_2 has nothing to do with p . By Remark 6.7, the place $K_2 \rightarrow K_3$ is defectless.

The place $K_1 \rightarrow K_2$ is defectless because it is spherically complete. To see this, note that $K \rightarrow K_3$ has the countable chains of balls condition of Remark 6.5. Therefore so does $K_1 \rightarrow K_2$. But the value group of $K_1 \rightarrow K_2$ is Δ/Δ_0 which is archimedean, hence has cofinality \aleph_0 . It follows that *any* chain of balls has non-empty intersection, so $K_1 \rightarrow K_2$ is spherically complete, which implies henselian+defectless.

Finally, the place $K_1 \rightarrow K_2$ is henselian defectless because it is equicharacteristic 0. So, each of the three places in (3) is henselian and defectless. Therefore their composition $K \rightarrow K_3$ is defectless, by Remark 6.5.

Now suppose that absolute ramification is unbounded. We first claim that Δ_0 is p -divisible. Indeed, by considering K^{sh} , one sees that K_2 is a NIP field, so the value group of $K_2 \rightarrow K_3$ must be p -divisible by Proposition 5.4 of [9].

Let Δ_p be the largest p -divisible convex subgroup of Γ . The group Δ_p is definable (in K), and it contains Δ_0 . By unbounded ramification, Δ_0 is *not* definable (in K), so Δ_p is strictly bigger than Δ_0 . Since Δ is the smallest convex group strictly bigger than Δ_0 , it follows that $\Delta_p \supseteq \Delta$ which means v is roughly p -divisible. \square

This Lemma is actually true if we replace “dp-rank 1” with “strongly dependent,” which is also preserved by Shelahification, and implies field perfection.

Now we complete the proof of Theorem 1.2.

Proof (of Theorem 1.2). Let K be a sufficiently saturated dp-minimal field.

We need to produce a valuation v satisfying the conditions of Theorem 1.1:

- For every n , $|\Gamma/n\Gamma|$ is finite.
- The residue field is algebraically closed, or elementarily equivalent to a local field of characteristic zero.
- The valuation is defectless
- The valuation is roughly p -divisible (Definition 6.4).

Any valuation on K will automatically satisfy the first condition, by Theorem 5.14.4, so we will henceforth ignore it.

Consider the valuation v_∞ from Theorem 6.2.

Case 1: The residue field Kv_∞ is a model of RCF or ACF_0 . In this case, we take $v = v_\infty$. Since the valuation is equicharacteristic 0, the defectlessness and rough p -divisibility conditions are automatic.

Case 2: The valuation v_∞ is definable and the residue field is finite. By [9] Proposition 5.3, K must have characteristic zero. Let Γ be the value group of v_∞ , let Δ be the smallest convex subgroup containing $v_\infty(p)$, and let Δ_0 be the largest convex subgroup avoiding $v_\infty(p)$. These groups are definable in K^{sh} , so there is a K^{sh} -interpretable factorization of the place $K \rightarrow Kv_\infty$:

$$K \xrightarrow{\Gamma/\Delta_0} K' \xrightarrow{\Delta_0} Kv_\infty$$

If K' is infinite, the place $K' \rightarrow Kv_\infty$ would violate Proposition 5.3 of [9], because $K' \rightarrow Kv_\infty$ is interpretable in the NIP structure K^{sh} . So K' is finite and $K' \rightarrow Kv_\infty$ is trivial, making Δ_0 be trivial.

As a consequence, $\Delta = \Delta/\Delta_0$, which is archimedean.

The convex subgroup Δ decomposes $K \rightarrow Kv_\infty$ as

$$K \xrightarrow{\Gamma/\Delta} K'' \xrightarrow{\Delta} Kv_\infty$$

By saturation, $K \rightarrow Kv_\infty$ satisfies the countable chains of balls condition of Remark 6.5. Therefore, so does $K'' \rightarrow Kv_\infty$, which has archimedean value group. Thus $K'' \rightarrow Kv_\infty$ is spherically complete.

Now $K'' \rightarrow Kv_\infty$ is a spherically complete field of characteristic zero, with finite residue field, and value group isomorphic to \mathbb{Z} . It follows that K'' is actually a local field. We take v to be the valuation corresponding to $K \rightarrow K''$ (i.e., v_∞ coarsened by Δ). As in Case 1, the defectlessness and rough p -divisibility conditions are automatic.

Case 3: The residue field of v_∞ is a model of ACF_p . In this case, we will take $v = v_\infty$.

Let v_1 be some definable valuation on K whose residue field Kv_1 has characteristic p . (If none such exists, then K has characteristic zero and $1/p \in \mathcal{O}$ for any definable valuation ring \mathcal{O} , so $1/p \in \mathcal{O}_\infty$ and v_∞ is equicharacteristic 0, a contradiction.) Note that the valuation v_1 might be trivial, and might be v_∞ .

The place $K \rightarrow Kv_\infty$ factors as $K \rightarrow Kv_1 \rightarrow Kv_\infty$ because v_∞ is finer than v_1 . By Remark 6.7, Kv_1 has nothing to do with p . Therefore, $Kv_1 \rightarrow Kv_\infty$ is roughly p -divisible and defectless.

By Remark 6.5, it remains to see that $K \rightarrow Kv_1$ is roughly p -divisible and defectless. By Lemma 6.8, it suffices to show that the mixed-characteristic valuation v_1 has unbounded ramification.

Suppose not. By Theorem 5.14.4, the p th-power map $K^\times \rightarrow K^\times$ has finite cokernel. Let \mathcal{O} and \mathfrak{m} denote the valuation ring and maximal ideal of v_1 . Applying the snake lemma to

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{O}^\times & \longrightarrow & K^\times & \longrightarrow & \Gamma \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathcal{O}^\times & \longrightarrow & K^\times & \longrightarrow & \Gamma \longrightarrow 1 \end{array}$$

and

$$\begin{array}{ccccccc}
1 & \longrightarrow & (1 + \mathfrak{m})^\times & \longrightarrow & \mathcal{O}^\times & \longrightarrow & Kv_1^\times \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & (1 + \mathfrak{m})^\times & \longrightarrow & \mathcal{O}^\times & \longrightarrow & Kv_1^\times \longrightarrow 1
\end{array}$$

where all the vertical maps are multiplication by p , we see that the p th power map on $(1 + \mathfrak{m})^\times$ also has finite cokernel.

If there was bounded ramification, then $\mathfrak{m} = (\tau)$ for some element τ of minimal v_1 -valuation. The p th power map on $1 + (\tau)$ lands in $1 + (\tau^p, p \cdot \tau) \subseteq 1 + \mathfrak{m}^2$.

However, $(1 + \mathfrak{m})^\times / (1 + \mathfrak{m}^2)^\times \cong \mathcal{O}/\mathfrak{m} \cong Kv_1$. So Kv_1 must be finite, which is absurd, because it has a place $Kv_1 \rightarrow Kv_\infty$ with algebraically closed residue field. \square

7 VC-minimal fields

In [4] Definition 1.4, Guingona makes the following definition:

Definition 7.1. A theory T is *dp-small* if there does not exist a model $M \models T$, formulas $\phi_i(x; y_i)$ with $|x| = 1$, and a formula $\psi(x; z)$, and elements a_{ij}, b_i, c_j such that

$$M \models \phi_i(a_{i'j}, b_i) \iff i = i'$$

$$M \models \psi(a_{ij'}, c_j) \iff j = j'$$

The sort of pattern here is more general than the one in the definition of dp-minimality, so dp-smallness is a stricter condition than dp-minimality.

Like dp-minimality, dp-smallness is preserved under reducts and under naming parameters. Guingona shows that VC-minimal fields are dp-small.

Theorem 7.2. Let K be a dp-small field. Then K is algebraically closed or real closed.

Proof. We may (and should) take K to be sufficiently saturated. By Theorem 1.6.4 of [4], the value group vK is divisible for any definable valuation v on K .

By Theorem 6.2, there is a henselian valuation v_∞ on K whose valuation ring is the intersection of all definable valuation rings on K . The residue field of v_∞ is algebraically closed, real closed, or finite. In the finite case, v_∞ is definable, and we saw in the proof of Theorem 1.2, that the value group of v_∞ has a least element, so the value group $v_\infty K$ is *not* divisible, a contradiction.

We must therefore be in case 1 or case 3 of the proof of Theorem 1.2. In particular, v_∞ is a henselian defectless valuation on K with an algebraically closed or real closed residue field. For K to be algebraically closed or real closed, it suffices to show that v_∞ has a divisible value group (by Ax-Kochen-Ershov in the real closed case, and defectlessness in the algebraically closed case).

Let ℓ be any prime. Let a be an element of K^\times . For each definable valuation \mathcal{O} on K , the value group $K^\times/\mathcal{O}^\times$ is ℓ -divisible. So there is an element $b \in K^\times$ and $c \in \mathcal{O}^\times$ such that $a = b^\ell \cdot c$. The valuation ring \mathcal{O}_∞ of v_∞ is the intersection of a small ordered set of \mathcal{O} 's, so by compactness, we can find $b \in K^\times$ and $c \in \mathcal{O}_\infty^\times$ such that $a = b^\ell \cdot c$. Then $v_\infty(a) = \ell \cdot v_\infty(b)$. So v_∞ has ℓ -divisible value group, for arbitrary ℓ . \square

8 Proof sketch of Theorem 1.1

Let (K, v) be a henselian defectless field with value group Γ and residue field k . Suppose that $\Gamma/n\Gamma$ is finite for all $n \in \mathbb{N}$. Suppose k is elementarily equivalent to a local field of characteristic 0, or $k \models ACF_p$, in which case $[-v(p), v(p)] \subset p \cdot \Gamma$.

Theorem 1.1 says two things:

- Completeness: the theory of (K, v) as a valued field is completely determined by the theories of k , Γ , and the type of $v(p)$ in the mixed characteristic case.
- Dp-minimality: (K, v) is dp-minimal as a valued field.

We give an outline of the proof, as a series of exercises. In what follows, an inclusion $(K, v) \hookrightarrow (L, v)$ of valued fields will be called *pure* if the inclusion of value groups is, i.e., $v(L)/v(K)$ is torsionless. “Definable” will mean “definable with parameters” unless specified otherwise.

1. Let $(\Gamma, +, <)$ be an ordered abelian group such that $\Gamma/n\Gamma$ is finite for all $n \in \mathbb{N}$. Show that Γ has quantifier elimination after adding unary predicates for *all* cuts and all sets of the form $\gamma_0 + n\Gamma$ ($\gamma_0 \in \Gamma$).
2. Let Γ be an ordered abelian group such that $\Gamma/n\Gamma$ is finite for all $n \in \mathbb{N}$. Show that every definable set in Γ is a boolean combination of sets of the form $\gamma_0 + n\Gamma$ and definable cuts.
3. Let (M, v) be a henselian field with residue characteristic p . Suppose M has nothing to do with p (Definition 6.6). Let K be a pure subfield that is perfect and henselian, and has separably closed residue field.
 - (a) Show that $\text{Gal}(K)$ is solvable.
 - (b) Show that $K^{alg} \cap M$ has nothing to do with p .
 - (c) Show that $K^{alg} \cap M$ is the fixed field of a Hall prime-to- p subgroup of $\text{Gal}(K)$.
4. Let T_Γ be a theory of p -divisible ordered abelian groups, having quantifier elimination in some relational language L_Γ . Let T be the theory of henselian defectless valued fields (K, v) with residue field modelling ACF_p , and with value group Γ . Show that T has quantifier elimination in the one-sorted language of valued fields expanded by all predicates of the form $R(v(x_1), \dots, v(x_n))$ where R is an n -ary predicate in L_Γ .
5. Prove the completeness part of Theorem 1.1 by combining 4 with Ax-Kochen-Ershov. (In the mixed characteristic case, coarsen by the largest p -divisible convex subgroup of Γ and use both AKE and 4.)
6. Suppose (M, v) is a κ -strongly homogeneous henselian valued field of residue characteristic 0. Suppose K and L are subfields of size less than κ , and $f : K \xrightarrow{\sim} L$ is an isomorphism of valued fields, such that the induced maps $v(K) \rightarrow v(L)$ and $\text{res}(K) \rightarrow \text{res}(L)$ are partial elementary maps on $v(M)$ and $\text{res}(M)$.
 - (a) Show that the inclusion $K \hookrightarrow M$ is pure if and only if $L \hookrightarrow M$ is pure.
 - (b) If $K \hookrightarrow M$ is pure, show that f extends to an automorphism of M (i.e., f is a partial elementary map).

7. Suppose (M, v) is henselian of residue characteristic 0. Suppose the n th power map $M^\times \rightarrow M^\times$ has finite cokernel for all $n \in \mathbb{N}$. Show that the following collection of sets generates all definable subsets of M under translations, rescalings, and boolean combinations:

- Macintyre predicates $(M^\times)^n$
- Sets of the form $\text{res}^{-1}(S)$ where S is a definable subset of $\text{res}(M)$.
- Sets of the form $v^{-1}(S)$ where S is a definable subset of $v(M)$.

Hint: reduce to the case where M is spherically complete and contains representatives from all cosets of $\bigcap_{n \in \mathbb{N}} (M^\times)^n$. Consider 1-types over M .

8. Suppose that (K, v) is henselian and defectless, $\text{res}(K) \models \text{ACF}_p$, and the value group $v(K)$ is p -divisible. Show that the definable subsets of K are generated under translations, rescalings, and boolean combinations, by sets of the form $v^{-1}(S)$ where S is a definable subset of $v(K)$. Hint: reduce to the case where K is spherically complete, and consider 1-types over K .
9. Let (K, v) be a henselian defectless valued field with algebraically closed residue field and value group Γ . Suppose $\Gamma/n\Gamma$ is finite for all n . If the residue characteristic is $p > 0$, suppose furthermore that $[v(p), -v(p)] \subseteq p \cdot \Gamma$. Show that every definable subset of K is a boolean combination of sets of the form

$$\{x : v(x - c) \in \Xi\}$$

where Ξ is a definable cut in Γ , and

$$\{x : v(b \cdot (x - c)) \in n \cdot \Gamma\}. \quad (4)$$

Hint: combine 7 and 8.

10. Let (K, v) be a henselian valued field with real-closed residue field, and value group Γ satisfying $\Gamma/n\Gamma$ finite for all n .
- (a) Show that K admits finitely many orderings, which are all definable.
- (b) Fix some ordering. Show that every definable subset of K is a boolean combination of sets of the form $\{x : x > c\}$,

$$\{x : v(x - c) \in \Xi\}$$

where Ξ is a definable cut in Γ , and

$$\{x : v(b \cdot (x - c)) \in n \cdot \Gamma\} \quad (5)$$

Hint: combine 7 with o-minimality of RCF.

11. Let (K, v) be a henselian valued field of mixed characteristic, with finite residue field and bounded absolute ramification. Let Γ be the value group, and suppose $\Gamma/n\Gamma$ is finite for all n . Show that every definable subset of K is a boolean combination of sets of the form

$$\{x : v(x - c) \in \Xi\} \quad (6)$$

where Ξ is a definable cut in Γ , and

$$\{x : P_n(b \cdot (x - c))\} \quad (7)$$

where P_n is the n th Macintyre predicate.

12. In the setting of 9 or 11, let L_0 be the language of valued fields expanded with all predicates of the form $v^{-1}(\Xi)$ for Ξ a definable cut in the value group. Show that (K, v) is dp-minimal on the level of quantifier-free L_0 formulas. More specifically, show that if b_1, b_2, \dots and c_1, c_2, \dots are mutually indiscernible sequences of tuples from K and $a \in K^1$, then either b_1, b_2, \dots or c_1, c_2, \dots is quantifier-free- L_0 -indiscernible over a . Hint: ACVF remains dp-minimal after naming all cuts in the value group.
13. In the setting of 10, let L_0 be the language of valued fields expanded with a binary predicate for one of the orderings, as well as all predicates of the form $v^{-1}(\Xi)$ for Ξ a definable cut in the value group. Show that (K, v) is dp-minimal on the level of quantifier-free L_0 formulas. More specifically, show that if b_1, b_2, \dots and c_1, c_2, \dots are mutually indiscernible sequences of tuples from K and $a \in K^1$, then either b_1, b_2, \dots or c_1, c_2, \dots is quantifier-free- L_0 -indiscernible over a . Hint: RCVF remains dp-minimal after naming all cuts in the value group.
14. In the setting of 9, 10, or 11, suppose (K, v) fails to be dp-minimal, and is sufficiently saturated. Show that there is an indiscernible sequence $\langle C_i \rangle_{i \in \mathbb{Z}}$ of sets of the form (4), (5), or (7), respectively, and an element $a \in K$ which belongs to some but not all of the C_i 's, such that the sequence $\langle c_i \rangle_{i \in \mathbb{Z}}$ of “centers” of the C_i 's is L_0 -quantifier-free indiscernible over a . (Hint: take a failure of dp-minimality. Write the sets explicitly in terms of the “cell-like” sets given in 9, 10, or 11, respectively. Arrange for the parameters to be mutually indiscernible and \mathbb{Z} -indexed. Let a be an element which is in the zeroth set in the first row and the zeroth set in the second row, but in no others. Use 12 or 13 to choose one of the two rows of sets whose parameter sequence is still L_0 -indiscernible over a . Using the explicit expression of the sets in the chosen row in terms of the “cell-like” sets, find “cell-like” C_i 's. By choice of L_0 , they must be of the form (4), (5), or (7), rather than $v^{-1}(\Xi)$ for Ξ a definable cut.)
15. In the setting of 14 plus 9 or 14 plus 10, argue that $v(a - c_i)$ depends on i . Using the L_0 -indiscernibility of the i 's over a , conclude that a is a pseudo-limit of the c_i 's as i goes to $+\infty$, or as i goes to $-\infty$. In the former case, argue that $v(a - c_i) = v(c_N - c_i)$ for $N > i$. Use full indiscernibility of the c_i 's over \emptyset to conclude that the coset

$$v(c_i - c_j) + \bigcap_{n \in \mathbb{N}} n\Gamma$$

doesn't depend on $i \neq j$. Conclude that $v(a - c_i) - v(a - c_j) \in \bigcap_{n \in \mathbb{N}} n\Gamma$ for all n and obtain a contradiction.

16. In the setting of 14 plus 11, argue that $rv_n(a - c_i)$ depends on i . Using the L_0 -indiscernibility of the i 's over a , argue that $rv_n(a - c_i) \neq rv_n(a - c_j)$ for $i \neq j$. Use the finiteness of the projection $rv_n(K) \rightarrow v(K)$ to argue that $v(a - c_i) \neq v(a - c_j)$

for $i \neq j$. As in 15 argue that the c_i pseudoconverge to a , perhaps after reversing their order. Conclude that $rv_n(a - c_i) = rv_n(c_N - c_i)$ for $N \gg i$. Use full indiscernibility of the c_i 's over \emptyset to conclude that the coset

$$rv_n(c_i - c_j) + \bigcap_{m \in \mathbb{N}} m \cdot rv_n(K)$$

doesn't depend on i, j . Conclude that $rv_n(a - c_i) - rv_n(a - c_j) \in \bigcap_{m \in \mathbb{N}} m \cdot rv_n(K)$ and obtain a contradiction.

17. If (K, v) is henselian with suitable value group and a residue field that is non-archimedean local of characteristic 0, prove that (K, v) is dp-minimal by viewing v as a coarsening of a valuation w such that (K, w) is as in 11.

We also remark that the equicharacteristic zero case of Theorem 1.1 has been proven in unpublished work by Chernikov and Simon: they show that an equicharacteristic 0 valued field with inp-minimal value group and residue field is itself inp-minimal.

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